

Exact solution for the energy density inside a one-dimensional non-static cavity with an arbitrary initial field state

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We study the exact solution for the energy density of a real massless scalar field in a two-dimensional spacetime, inside a non-static cavity with an arbitrary initial field state, taking into account the Neumann and Dirichlet boundary conditions. This work generalizes the exact solution proposed by Cole and Schieve in the context of the Dirichlet boundary condition and vacuum as the initial state. We investigate diagonal states, examining the vacuum and thermal field as particular cases. We also study non-diagonal initial field states, taking as examples the coherent and Schrödinger cat states.

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I. INTRODUCTION

The first works investigating the quantum problem of the radiation generated by moving mirrors in vacuum were published in the 1970s decade (see Refs. [1, 2, 3, 4]). Moore [1], in the context of a real massless scalar field in a two dimensional spacetime, investigated the radiation generated in a cavity with a moving boundary. Imposing Dirichlet boundary condition to the field and also a prescribed law for the movement of a boundary, Moore obtained an exact formula for the expected value of the energy-momentum tensor, assuming the initial field state as the vacuum. The field solution obtained by Moore is given in terms of a functional equation, usually called Moore's equation (which was obtained independently by Vesnitskii [5]), for which there is no general technique of analytical solution. Law [6] obtained an exact analytic solution for the Moore equation for a particular resonant movement of the boundary (see also Ref. [7]). Cole and Schieve [8] proposed a numerical method to solve exactly the Moore equation for a general law of motion of the boundary. Approximate analytical solutions of the Moore equation were also obtained, for instance, by Dodonov-Klimov-Nikonov [9] and Dalvit-Mazzitelli [10]. With different approaches from those adopted by Moore [1] and Fulling-Davies [2], perturbative methods were developed to solve the problem of a quantum field in the presence of a single moving boundary [11] and also in oscillating cavities [12].

The first works studying the problem of particle creation by moving mirrors with initial states different from vacuum were also published about thirty years ago [3], showing that the presence of real particles in the initial state amplifies the phenomenon of particle creation. Considering a thermal bath as the “in” field state, the dynamical Casimir effect has been investigated for the case of a single mirror [13, 14, 15], and also for an oscillating cavity [9, 14, 16]. The coherent state as initial field state has been considered for one moving boundary in Refs. [15, 17, 18, 19], and the superposition of coherent states in connection to the investigation of decoherence via the

dynamical Casimir effect [20]. Squeezed states have also been considered by three of the present authors [15].

Recently, several works have also investigated the influence of different boundary conditions on the dynamical Casimir effect [15, 17, 21, 22]. In this context, it was showed that, for the single mirror problem, Dirichlet and Neumann boundary conditions yield the same energy density radiated when the initial field state is symmetrical under time translations [15, 17].

In the present work we investigate the time evolution of the energy density for a real massless scalar field in a two-dimensional spacetime, inside a non-static cavity, considering both the Dirichlet and Neumann boundary conditions. We extend to an arbitrary initial field state the exact method for calculating the energy density proposed by Cole and Schieve [8, 23] in the context of the Dirichlet boundary condition and vacuum as the initial state. Considering an arbitrary initial field state, we show that the energy density in a given point of the spacetime can be obtained by tracing back a sequence of null lines, connecting the value of the energy density at the given spacetime point to a certain known value of the energy at a point in the “static zone”, where the initial field modes are not affected by the perturbation caused by the boundary motion. We investigate diagonal states (for which the static energy density is invariant under time translation), examining in particular the vacuum and thermal field states. We also study non-diagonal initial field states (in this case the static energy density is not invariant under time translation), investigating the coherent and the Schrödinger cat states.

The paper is organized as follows. In Sec. II, considering Neumann and Dirichlet boundary conditions and also an arbitrary initial field state, we write the field solution and the energy density in the non-static cavity, generalizing the formula found in the literature [1, 2], which is valid for the Dirichlet boundary condition and vacuum as the initial field state. In Sec. III we discuss the energy density in the static situation. In Sec. IV we obtain the energy density in the non-static situation given in terms of the static energy density. In Sec. V we summarize our

main results.

II. ENERGY DENSITY: EXACT FORMULAS

Let us start considering the field satisfying the Klein-Gordon equation (we assume throughout this paper $\hbar = c = k_B = 1$): $(\partial_t^2 - \partial_x^2)\psi(t, x) = 0$, and obeying conditions imposed at the static boundary located at $x = 0$, and also at the moving boundary's position at $x = L(t)$, where $L(t)$ is a prescribed law for the moving boundary with $L(t < 0) = L_0$, where L_0 is the length of the cavity in the static situation. We consider four types of boundary conditions. The Dirichlet-Neumann (DN) boundary condition imposes Dirichlet condition at the static boundary, whereas the space derivative of the field taken in the instantaneously co-moving Lorentz frame vanishes at the moving boundary's position (Neumann condition): $\partial_{x'}\psi'(t', x')|_{\text{boundary}} = 0$. Using the appropriate Lorentz transformation, this boundary condition can be written in terms of quantities in the laboratory inertial frame as follows: $\partial_{x'}\psi'(t', x') = \left[\left(\dot{L}(t)\partial_t + \partial_x\right)\psi(t, x)\right]_{x=L(t)} = 0$. The Neumann-Neumann (NN) boundary condition imposes $[\partial_x\psi(t, x)]_{x=0} = 0$ and $\left[\left(\dot{L}(t)\partial_t + \partial_x\right)\psi(t, x)\right]_{x=L(t)} = 0$. The Neumann-Dirichlet (ND) boundary condition imposes $[\partial_x\psi(t, x)]_{x=0} = 0$ and $\psi(t, L(t)) = 0$, whereas the Dirichlet-Dirichlet (DD) boundary condition imposes $\psi(t, 0) = 0$, and $\psi(t, L(t)) = 0$. Considering the procedure adopted in Refs. [1, 2], the field in the cavity can be obtained by exploiting the conformal invariance of the Klein-Gordon equation. The field operator, solution of the wave equation, is given by:

$$\hat{\psi}(t, x) = \lambda[\hat{A} + \hat{B}\psi^{(0)}(t, x)] + \sum_{n=1-2\beta}^{\infty} [\hat{a}_n\psi_n(t, x) + H.c.], \quad (1)$$

where $\psi^{(0)} = [R(v) + R(u)]/2$ [24], and the field modes $\psi_n(t, x)$ are given by:

$$\psi_n(t, x) = \frac{1}{\sqrt{4(n+\beta)\pi}} \left[\gamma \varphi_n^{(\beta)}(v) + \gamma^* \varphi_n^{(\beta)}(u) \right], \quad (2)$$

with $\varphi_n^{(\beta)}(z) = e^{-i(n+\beta)\pi R(z)}$, $u = t - x$, $v = t + x$, and R satisfying Moore's functional equation:

$$R[t + L(t)] - R[t - L(t)] = 2. \quad (3)$$

The operators \hat{A} and \hat{B} satisfy the commutation rules $[\hat{A}, \hat{B}] = i$, $[\hat{A}, \hat{a}_n] = [\hat{B}, \hat{a}_n] = 0$ [24]. In Eqs. (1) and (2) we introduce a notation which enables us to put into a single formula the solutions for the four boundary conditions considered in the present work. In this sense, for $\lambda = \gamma = 1$ and $\beta = 0$, Eqs. (1) and (2) give the NN solution. The other three possible situations are recovered if we consider $\lambda = 0$ and: $\beta = 0$ and $\gamma = i$ for the

DD case; $\beta = 1/2$ and $\gamma = i$ for the DN case; $\beta = 1/2$ and $\gamma = 1$ for the ND case.

Hereafter, considering the Heisenberg picture, we are interested in the averages $\langle \dots \rangle$ taken over any initial field state annihilated by \hat{B} . In this context, we will write the exact formulas for the expected value of the energy density operator $\mathcal{T} = \langle \hat{T}_{00}(t, x) \rangle$. We can split \mathcal{T} , writing:

$$\mathcal{T} = \mathcal{T}_{\text{vac}} + \mathcal{T}_{\text{non-vac}}, \quad (4)$$

where

$$\mathcal{T}_{\text{vac}} = \frac{\pi|\gamma|^2}{4} \sum_{n=1-2\beta}^{\infty} (n+\beta) \left[R'^2(v) + R'^2(u) \right] \quad (5)$$

and

$$\mathcal{T}_{\text{non-vac}} = \mathcal{T}_{\langle \hat{a}^\dagger \hat{a} \rangle} + \mathcal{T}_{\langle \hat{a} \hat{a} \rangle}, \quad (6)$$

with

$$\mathcal{T}_{\langle \hat{a}^\dagger \hat{a} \rangle} = g_1(v) + g_1(u), \quad (7)$$

$$\mathcal{T}_{\langle \hat{a} \hat{a} \rangle} = g_2(v) + g_2(u), \quad (8)$$

$$g_1(z) = \frac{\pi|\gamma|^2}{2} \sum_{n, n'=1-2\beta}^{\infty} \sqrt{(n+\beta)(n'+\beta)} \times \text{Re} \left\{ e^{i(n-n')\pi R(z)} [R'(z)]^2 \langle \hat{a}_n^\dagger \hat{a}_{n'} \rangle \right\}, \quad (9)$$

$$g_2(z) = -\frac{\pi\gamma^2}{2} \sum_{n, n'=1-2\beta}^{\infty} \sqrt{(n+\beta)(n'+\beta)} \times \text{Re} \left\{ e^{-i(n+n'+2\beta)\pi R(z)} [R'(z)]^2 \times \langle \hat{a}_n \hat{a}_{n'} \rangle \right\}. \quad (10)$$

The term \mathcal{T}_{vac} is the local energy density related to the vacuum state, which is divergent. Following Ref. [2], we adopt the point-splitting regularization method and obtain \mathcal{T}_{vac} (now redefined) as the renormalized local energy density:

$$\mathcal{T}_{\text{vac}} = -f(v) - f(u), \quad (11)$$

where

$$f = \frac{|\gamma|^2}{24\pi} \left\{ \frac{R'''}{R'} - \frac{3}{2} \left(\frac{R''}{R'} \right)^2 + \pi^2 \left[\frac{1}{2} - 3(\beta - \beta^2) \right] R'^2 \right\}. \quad (12)$$

In the above equation the derivatives are taken with respect to the argument of the R function.

We can also write the non-vacuum part of the energy density in the analogous notation:

$$\mathcal{T}_{\text{non-vac}} = -g(v) - g(u), \quad (13)$$

where $g = -(g_1 + g_2)$. Note that \mathcal{T}_{vac} and $\mathcal{T}_{\langle\hat{a}^\dagger\hat{a}\rangle}$ depend on $|\gamma|^2$, which has the same value for the considered boundary conditions. On the other hand $\mathcal{T}_{\langle\hat{a}\hat{a}\rangle}$ depends on γ^2 , which can differ by a sign, depending on the situation. For further analysis, it is useful to write:

$$\mathcal{T} = -h(v) - h(u), \quad (14)$$

where

$$h = f + g. \quad (15)$$

III. STATIC SITUATION

Let us first examine the formulas in the preceding section for the static situation $t \leq 0$, when the Moore equation is reduced to $R(t + L_0) - R(t - L_0) = 2$. For this case the function R is given by [1]:

$$R(z) = z/L_0. \quad (16)$$

The functions f , g , g_1 and g_2 , now relabeled, respectively, as $f^{(s)}$, $g^{(s)}$, $g_1^{(s)}$ and $g_2^{(s)}$, are given by:

$$f^{(s)} = \frac{|\gamma|^2 \pi}{24L_0^2} \left[\frac{1}{2} - 3(\beta - \beta^2) \right], \quad (17)$$

$$g^{(s)} = -(g_1^{(s)} + g_2^{(s)}), \quad (18)$$

$$g_1^{(s)}(z) = \frac{\pi |\gamma|^2}{2L_0^2} \sum_{n,n'=1-2\beta}^{\infty} \sqrt{(n+\beta)(n'+\beta)} \times \text{Re} \left\{ e^{i(n-n')\pi z/L_0} \langle \hat{a}_n^\dagger \hat{a}_{n'} \rangle \right\}, \quad (19)$$

$$g_2^{(s)}(z) = -\frac{\pi \gamma^2}{2L_0^2} \sum_{n,n'=1-2\beta}^{\infty} \sqrt{(n+\beta)(n'+\beta)} \times \text{Re} \left\{ e^{-i(n+n'+2\beta)\pi z/L_0} \langle \hat{a}_n \hat{a}_{n'} \rangle \right\}. \quad (20)$$

Note that $f^{(s)}$ is a constant and \mathcal{T}_{vac} (in this case the Casimir energy density and relabeled as \mathcal{T}_{cas}) is such that (see Refs. [24, 25]):

$$\mathcal{T}_{\text{cas}}^{\text{DD}} = \mathcal{T}_{\text{cas}}^{\text{NN}} = -\frac{\pi}{24L_0^2}, \quad \mathcal{T}_{\text{cas}}^{\text{DN}} = \mathcal{T}_{\text{cas}}^{\text{ND}} = \frac{\pi}{48L_0^2}, \quad (21)$$

where the superscripts DD, NN, DN and ND mean the types of boundary conditions considered in the calculations. Note that, for mixed boundary conditions (in this case ND or DN), the Casimir energy is positive, originating a repulsive Casimir force (see Ref. [26]). On the other hand, $g^{(s)}$ is, in general, spacetime dependent, what implies that $\mathcal{T}_{\langle\hat{a}^\dagger\hat{a}\rangle}$ and $\mathcal{T}_{\langle\hat{a}\hat{a}\rangle}$, in this static situation relabeled respectively as $\mathcal{T}_{\langle\hat{a}^\dagger\hat{a}\rangle}^{(s)}$ and $\mathcal{T}_{\langle\hat{a}\hat{a}\rangle}^{(s)}$, are functions of

the spacetime variables. Despite the fact that $\mathcal{T}_{\langle\hat{a}^\dagger\hat{a}\rangle}^{(s)}$ and $\mathcal{T}_{\langle\hat{a}\hat{a}\rangle}^{(s)}$ can depend on time, from the principle of energy conservation the total energy is a constant in time for the static situation. In fact, for an arbitrary initial field state, the integration of Eqs. (19) and (20) yields:

$$\int_0^{L_0} \mathcal{T}_{\langle\hat{a}^\dagger\hat{a}\rangle}^{(s)} dx = \sum_{n=1-2\beta}^{\infty} \omega_n \mathcal{N}_n, \quad (22)$$

$$\int_0^{L_0} \mathcal{T}_{\langle\hat{a}\hat{a}\rangle}^{(s)} dx = 0, \quad (23)$$

where $\omega_n = \pi(n + \beta)/L_0$, and $\mathcal{N}_n = \langle \hat{a}_n^\dagger \hat{a}_n \rangle$ is the initial mean number of particles in the n th mode. Then, for the static cavity, the function $\mathcal{T}_{\langle\hat{a}\hat{a}\rangle}^{(s)}$ can give contribution for the spacetime behavior of the energy density \mathcal{T} (now relabeled as $\mathcal{T}^{(s)}$), but not for the total energy stored in the cavity. The total static energy $\mathcal{E}^{(s)} = \int_0^{L_0} \mathcal{T}^{(s)} dx$ can be written as $\mathcal{E}^{(s)} = \mathcal{E}_{\text{cas}} + \sum_n \omega_n \mathcal{N}_n$, where $\mathcal{E}_{\text{cas}} = \int_0^{L_0} \mathcal{T}_{\text{cas}} dx$. Observe that, for a static cavity, for any initial field state considered, we have: $\mathcal{E}^{(s)\text{ND}} = \mathcal{E}^{(s)\text{DN}}$ and $\mathcal{E}^{(s)\text{DD}} = \mathcal{E}^{(s)\text{NN}}$.

For the initial field states such that the density matrix is diagonal in the Fock basis, so that $\langle \hat{a}_n \hat{a}_{n'} \rangle = 0$ and $\langle \hat{a}_n^\dagger \hat{a}_{n'} \rangle = \mathcal{N}_n \delta_{nn'}$ [19], we have:

$$g_2^{(s)}(z) = 0, \quad \partial_z[g_1^{(s)}(z)] = 0, \quad \partial_z[h^{(s)}(z)] = 0, \quad g^{(s)} \leq 0, \quad (24)$$

leading to an uniform energy density in the static zone, which is invariant under time translation. For this case we have:

$$\mathcal{T}^{(s)\text{DD}} = \mathcal{T}^{(s)\text{NN}}, \quad \mathcal{T}^{(s)\text{DN}} = \mathcal{T}^{(s)\text{ND}}. \quad (25)$$

This extends the equalities $\mathcal{T}_{\text{cas}}^{\text{DD}} = \mathcal{T}_{\text{cas}}^{\text{NN}}$ and $\mathcal{T}_{\text{cas}}^{\text{DD}} = \mathcal{T}_{\text{cas}}^{\text{NN}}$, showed in Eqs. (21), from the vacuum to any other diagonal field state.

For initial field states with the density matrix having nonzero off-diagonal elements, the function $g^{(s)}$ can have spacetime dependence, what leads to an energy density non-invariant under time translation. For two cavities with different boundary conditions, but the same length L_0 and values for $\langle \hat{a}_n^\dagger \hat{a}_{n'} \rangle$ we have:

$$\mathcal{T}_{\langle\hat{a}^\dagger\hat{a}\rangle}^{(s)\text{DD}} = \mathcal{T}_{\langle\hat{a}^\dagger\hat{a}\rangle}^{(s)\text{NN}}, \quad \mathcal{T}_{\langle\hat{a}^\dagger\hat{a}\rangle}^{(s)\text{DN}} = \mathcal{T}_{\langle\hat{a}^\dagger\hat{a}\rangle}^{(s)\text{ND}}. \quad (26)$$

On the other hand, for the same values of $\langle \hat{a}_n \hat{a}_{n'} \rangle$, we get:

$$\mathcal{T}_{\langle\hat{a}\hat{a}\rangle}^{(s)\text{DD}} = -\mathcal{T}_{\langle\hat{a}\hat{a}\rangle}^{(s)\text{NN}}, \quad \mathcal{T}_{\langle\hat{a}\hat{a}\rangle}^{(s)\text{DN}} = -\mathcal{T}_{\langle\hat{a}\hat{a}\rangle}^{(s)\text{ND}}. \quad (27)$$

However, different and suitable choices for $\langle \hat{a}_n \hat{a}_{n'} \rangle$, for instance $\langle \hat{a}_n \hat{a}_{n'} \rangle_1$ and $\langle \hat{a}_n \hat{a}_{n'} \rangle_2$ can produce:

$$\mathcal{T}_{\langle\hat{a}\hat{a}\rangle_1}^{(s)\text{DD}} = \mathcal{T}_{\langle\hat{a}\hat{a}\rangle_2}^{(s)\text{NN}}, \quad \mathcal{T}_{\langle\hat{a}\hat{a}\rangle_1}^{(s)\text{DN}} = \mathcal{T}_{\langle\hat{a}\hat{a}\rangle_2}^{(s)\text{ND}}, \quad (28)$$

as showed in Sec. IV B, Eq. (59).

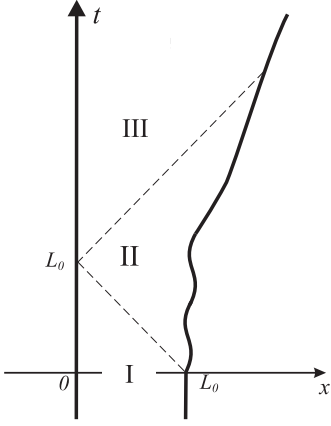


FIG. 1: Boundary trajectories (solid lines). The dashed lines are null-lines separating the regions I from II, and II from III.

IV. NON-STATIC SITUATION

Now, let us examine the cavity in the non-static situation ($t > 0$). The field modes in Eq. (1) are formed by left and right-propagating parts. As causality requires, the field in the region I ($v \leq L_0$) (see Fig. 1) is not affected by the boundary motion, so that, in this sense, this region is considered as a “static zone”. In the region II ($v > L_0$ and $u \leq L_0$), the right-propagating parts of the field modes remain unaffected by the boundary motion, so that the region II is also a static zone for these modes. On the other hand, the left-propagating parts in the region II are, in general, affected by the boundary movement. In the region III ($u > L_0$), both the left and right-propagating parts are affected. In summary, the functions corresponding to the left and right-propagating parts of the field modes are considered in the static zone if their argument z (z symbolizing v or u) is such that $z \leq L_0$. We remark that Eqs. (16), (17), (19) and (20) are valid not only in the static situation ($t \leq 0$) but also in the static zone ($t \pm x \leq L_0$).

For a certain spacetime point (\tilde{t}, \tilde{x}) , the field operator $\hat{\psi}(\tilde{t}, \tilde{x})$ is known if its left and right-propagating parts, taken over, respectively, the null lines $v = z_1$ and $u = z_2$ (where $z_1 = \tilde{t} + \tilde{x}$ and $z_2 = \tilde{t} - \tilde{x}$), are known; or, in other words, $\hat{\psi}(\tilde{t}, \tilde{x})$ is known if $R(v)|_{v=z_1}$ and $R(u)|_{u=z_2}$ are known. Cole and Schieve [8] proposed a recursive method to obtain the function R for a general law of motion of the boundary, tracing back a sequence of null lines until a null line gets into the static zone where the R function is known. For a brief discussion of this method, let us assume that (\tilde{t}, \tilde{x}) belongs to the region III, and that the null line $v = z_1$ intersects the moving mirror trajectory at the point $[t_1, L(t_1)]$ (see Ref. [8]). We have $R(v)|_{v=z_1} = R[t_1 + L(t_1)]$. Using the Moore equation (3), we get $R(v)|_{v=z_1} = R(u)|_{u=t_1-L(t_1)} + 2$, so that, as pointed by Cole and Schieve, “the value of R increases by 2 every time there is a reflection off the moving wall” [8]. If $t_1 - L(t_1) \leq L_0$, then the null line

$u = t_1 - L(t_1)$ is already in the static zone, so that we can write $R(u)|_{u=t_1-L(t_1)} = [t_1 - L(t_1)]/L_0$, and also $R(v)|_{v=z_1} = [t_1 - L(t_1)]/L_0 + 2$. On the other hand, if $t_1 - L(t_1) > L_0$, we can draw another null line $v = t_1 - L(t_1)$ intersecting the world line of the static boundary at the point $[t_1 - L(t_1), 0]$. Then we get $R(u)|_{u=t_1-L(t_1)} = R(v)|_{v=t_1-L(t_1)}$, and can write $R(v)|_{v=z_1} = R(v)|_{v=t_1-L(t_1)} + 2$. Assuming that the null line $v = t_1 - L(t_1)$ intersects the world line of the moving boundary at the point $[t_2, L(t_2)]$, we get $t_2 + L(t_2) = t_1 - L(t_1)$, and we can write: $R(v)|_{v=z_1} = R(v)|_{v=t_2+L(t_2)} + 2$. In the same way, considering the Moore equation, we can write $R(v)|_{v=z_1} = R(u)|_{u=t_2-L(t_2)} + 4$, and so on, up to the null line considered is in the static zone, where the function R is known. In summary, we can write: $R(\tilde{z}) = 2n + [\tilde{z} - 2 \sum_{i=1}^n L(t_i)]/L_0$ [8], where n is the number of reflections off the moving boundary, necessary to connect the null line $v = \tilde{z}$ (or $u = \tilde{z}$) to a null line in the static zone.

It is interesting to notice the behavior of the left and right-propagating parts of the field modes in the cavity. Taking into account the above analysis for the R behavior, for DD and NN cases ($\beta = 0$) we get: $\varphi_n^{(0)}(v)|_{v=z_1} = \varphi_n^{(0)}(u)|_{u=t_1-L(t_1)}$, $\varphi_n^{(0)}(u)|_{u=t_1-L(t_1)} = \varphi_n^{(0)}(v)|_{v=t_1-L(t_1)}$, and $\varphi_n^{(0)}(v)|_{v=t_1-L(t_1)} = \varphi_n^{(0)}(u)|_{u=t_2-L(t_2)}$. We can see that the value of $\varphi_n^{(0)}$ does not change under a reflection off the static or moving boundaries. The relevant fact is the sequence of the spacetime points where the reflections occur, because it will map an initial null line in a non-static zone, into a certain null line in the static zone, where the value of $\varphi_n^{(0)}$ is determined. This last comment is also valid for mixed boundary conditions ($\beta = 1/2$), but, in this case, we observe that the value of $\varphi_n^{(1/2)}$ does not change under a reflection off the static boundary, but changes sign under a reflection off the moving boundary.

The behavior of the energy density in a cavity, for the vacuum as the initial state, is described by the function f , which obeys the following equation proposed by Cole and Schieve [23]:

$$f[t + L(t)] = f[t - L(t)]\mathcal{A}(t) + \mathcal{B}(t), \quad (29)$$

where

$$\mathcal{A}(t) = \left[\frac{1 - L'(t)}{1 + L'(t)} \right]^2, \quad (30)$$

$$\begin{aligned} \mathcal{B}(t) = & -\frac{1}{12\pi} \frac{L'''(t)}{[1 + L'(t)]^3 [1 - L'(t)]} \\ & -\frac{1}{4\pi} \frac{L''^2(t) L'(t)}{[1 + L'(t)]^4 [1 - L'(t)]^2}, \end{aligned} \quad (31)$$

which enable us to obtain $f(z)$, and therefore \mathcal{T}_{vac} , recursively in terms of its value $f = f^{(s)}$ in the static zone (see Refs. [8, 23]).

For an arbitrary initial field state, the behavior of the energy density in a cavity is described by the function

f (related to the vacuum part) and a new function g (related to the non-vacuum part) proposed in the present paper. For the latter one, from Eqs. (3), (9) and (10) we obtain the following equation:

$$g[t + L(t)] = g[t - L(t)]\mathcal{A}(t), \quad (32)$$

which enable us to get $g(z)$, and therefore $\mathcal{T}_{\text{non-vac}}$, recursively in terms of its value $g(z) = g^{(s)}(z)$ in the static zone.

Now, for a complete analysis involving vacuum and non-vacuum parts, let us write the equation for the function h (Eq. (14)) as follows:

$$h[t + L(t)] = h[t - L(t)]\mathcal{A}(t) + \mathcal{B}(t). \quad (33)$$

The procedure to find $h(z)$, solving recursively the Eq. (33), starts by setting $z = t_1 + L(t_1)$ and tracing the null line $v = t_1 + L(t_1)$. From Eq. (33), after the first reflection traced back on the moving boundary, we get the relation $h(z) = h[t_1 - L(t_1)]\mathcal{A}(t_1) + \mathcal{B}(t_1)$. Now, as explained above, we set $t_1 - L(t_1) = t_2 + L(t_2)$, and, after the second reflection on the moving boundary, we get:

$$h(z) = h[t_2 - L(t_2)]\mathcal{A}(t_2)\mathcal{A}(t_1) + \mathcal{B}(t_2)\mathcal{A}(t_1) + \mathcal{B}(t_1).$$

This process goes on, up to a null line gets into a static zone, where $h = h^{(s)} = f^{(s)} + g^{(s)}$ is known. At the end, we get:

$$h(z) = h^{(s)}(z)\tilde{\mathcal{A}}(z) + \tilde{\mathcal{B}}(z), \quad (34)$$

where:

$$h^{(s)}(z) = f^{(s)} + g^{(s)}[\tilde{z}(z)], \quad (35)$$

$$\tilde{\mathcal{A}}(z) = \Pi_{i=1}^n \mathcal{A}(t_i), \quad (36)$$

$$\begin{aligned} \tilde{\mathcal{B}}(z) = & \mathcal{B}(t_n)\Pi_{i=1}^{n-1}\mathcal{A}(t_i) + \mathcal{B}(t_{n-1})\Pi_{i=1}^{n-2}\mathcal{A}(t_i) \\ & + \dots + \mathcal{B}(t_2)\mathcal{A}(t_1) + \mathcal{B}(t_1). \end{aligned} \quad (37)$$

The number n of reflections and the sequence of instants t_1, \dots, t_n depend on z . In Eq. (35), we have $\tilde{z}(z) = t_n - L(t_n)$. From this process, we observe that the value of f does not change under a reflection off the static boundary, but it changes under a reflection off the moving boundary, as expected. We also note, from Eq. (33), that the function h , after a reflection, changes in the same way for all boundary conditions considered in the present work (for the same boundary motion). We remark that, in Eq. (34), the functions $\tilde{\mathcal{A}}$ and $\tilde{\mathcal{B}}$ depend only on the law of motion of the moving mirror, whereas all the dependence on the boundary conditions considered in the present paper, and on the initial field state, are stored in the static region function $h^{(s)}(z)$. We also observe that, for a generic law of motion, $\tilde{\mathcal{A}}$ and $\tilde{\mathcal{B}}$ are different functions, with the following properties:

$$\tilde{\mathcal{A}}(z) > 0 \forall z, \tilde{\mathcal{A}}(z < L_0) = 1, \tilde{\mathcal{B}}(z < L_0) = 0, \quad (38)$$

which are directly obtained from Eqs. (30) and (31).

The energy density \mathcal{T} can be given as in Eq. (4), but now with the parts \mathcal{T}_{vac} and $\mathcal{T}_{\text{non-vac}}$ rewritten as:

$$\mathcal{T}_{\text{vac}} = -f^{(s)}[\tilde{\mathcal{A}}(u) + \tilde{\mathcal{A}}(v)] - \tilde{\mathcal{B}}(u) - \tilde{\mathcal{B}}(v), \quad (39)$$

$$\mathcal{T}_{\text{non-vac}} = -g^{(s)}[\tilde{z}(u)]\tilde{\mathcal{A}}(u) - g^{(s)}[\tilde{z}(v)]\tilde{\mathcal{A}}(v). \quad (40)$$

The instantaneous force acting on the moving boundary, in this two-dimensional model, is given by $\mathcal{T}(t, L(t))$. From Eqs. (14), (39) and (40) we see that two cavities with different boundary conditions and the same initial length L_0 , presenting different vacuum and non-vacuum contributions for their initial energy densities, but with the same total energy density $\mathcal{T}^{(s)}$ in the static zone, for a same (but arbitrary) law of motion will always exhibit identical evolutions in time for their energy densities \mathcal{T} .

A. Diagonal states

Now, let us analyze and compare the behavior of \mathcal{T}_{vac} , $\mathcal{T}_{\text{non-vac}}$ and \mathcal{T} for the initial field states for which the density matrix is diagonal in the Fock basis. From Eqs. (24) and (40) we obtain:

$$\mathcal{T}_{\text{non-vac}} = -g^{(s)}[\tilde{\mathcal{A}}(u) + \tilde{\mathcal{A}}(v)], \quad (41)$$

where $g^{(s)}$ is a constant factor. From Eqs. (39) and (41) we can see that all spacetime dependence in \mathcal{T} comes from $\tilde{\mathcal{A}}$ and $\tilde{\mathcal{B}}$. For this case we have:

$$\mathcal{T}^{\text{DD}} = \mathcal{T}^{\text{NN}}, \mathcal{T}^{\text{DN}} = \mathcal{T}^{\text{ND}}. \quad (42)$$

Since Eq. (42) is valid for any law of motion, it generalizes the Eq. (25), and because it is also valid for any diagonal state, it extends the conclusion obtained in Ref. [22], where formulas correspondent to Eq. (42) were obtained just considering the vacuum as the initial field state.

The total energy stored in the cavity as a function of time can be written as $\mathcal{E}(t) = \int_0^{L(t)} \mathcal{T}(t, x) dx$. For diagonal initial states, we have:

$$\mathcal{E}(t) = -h^{(s)}\mathcal{F}_1(t) - \mathcal{F}_2(t), \quad (43)$$

where $\mathcal{F}_1(t) = \int_0^{L(t)} [\tilde{\mathcal{A}}(u) + \tilde{\mathcal{A}}(v)] dx$ and $\mathcal{F}_2(t) = \int_0^{L(t)} [\tilde{\mathcal{B}}(u) + \tilde{\mathcal{B}}(v)] dx$. We emphasize that \mathcal{F}_1 and \mathcal{F}_2 depend only on the law of motion of the boundary, whereas all dependence on the initial field state or type of boundary condition is stored in the coefficient $h^{(s)}$.

Since the functions $\tilde{\mathcal{A}}$ and $\tilde{\mathcal{B}}$ are, in general, different one from the other, we conclude that \mathcal{T}_{vac} , $\mathcal{T}_{\text{non-vac}}$ and \mathcal{T} exhibit, in general, different structures. On the other hand, one could ask about the conditions under which \mathcal{T}_{vac} , $\mathcal{T}_{\text{non-vac}}$ and \mathcal{T} would exhibit the same structure

(the same curve apart from a positive scale factor or an additive constant) so that, for instance, if there are peaks and valleys they would be at the same positions in the cavity. One condition is provided by laws of motion for which $\tilde{\mathcal{B}}(z)$ and $\tilde{\mathcal{A}}(z)$ have a linear relation as given by:

$$\tilde{\mathcal{B}}(z) = k_1 \tilde{\mathcal{A}}(z) + k_2, \quad (44)$$

where k_1 e k_2 are constants. From the properties in Eq. (38), we get $k_1 = -k_2$, resulting:

$$\tilde{\mathcal{B}}(z) = k_1 [\tilde{\mathcal{A}}(z) - 1]. \quad (45)$$

From (39), (40) and (45), we get:

$$\mathcal{T}_{\text{vac}} = -(f^{(s)} + k_1)[\tilde{\mathcal{A}}(u) + \tilde{\mathcal{A}}(v)] + 2k_1, \quad (46)$$

$$\mathcal{T}_{\text{non-vac}} = -g^{(s)}[\tilde{\mathcal{A}}(u) + \tilde{\mathcal{A}}(v)], \quad (47)$$

$$\mathcal{T} = -(f^{(s)} + g^{(s)} + k_1)[\tilde{\mathcal{A}}(u) + \tilde{\mathcal{A}}(v)] + 2k_1. \quad (48)$$

In the energy densities showed in Eqs. (46)-(48), if the constant factors multiplying $\tilde{\mathcal{A}}(u) + \tilde{\mathcal{A}}(v)$ are different from zero and have the same sign, as the ratio $\sigma(z) = \tilde{\mathcal{B}}(z)/[\tilde{\mathcal{A}}(z) - 1]$ becomes more approximately equal to a constant value k_1 , more the structures of \mathcal{T}_{vac} , $\mathcal{T}_{\text{non-vac}}$ and \mathcal{T} become similar one to the other. Next, we exemplify these situations in the context of the vacuum and thermal field, which are examples of a diagonal initial field states.

For the thermal state we take into account that $\langle \hat{a}_n^\dagger \hat{a}_{n'} \rangle = \delta_{nn'} \bar{n}(n, \beta)$ and $\langle \hat{a}_n \hat{a}_{n'}^\dagger \rangle = \langle \hat{a}_n^\dagger \hat{a}_{n'}^\dagger \rangle = 0$, where $\bar{n}(n, \beta) = 1/(e^{(n+\beta)/T} - 1)$, with the Boltzmann constant (k_B) equal to 1. For this initial state, the energy density $\mathcal{T}_{\text{non-vac}}$, now relabeled as \mathcal{T}_T , is given by:

$$\mathcal{T}_T = \frac{\pi |\gamma|^2}{2L_0^2} \mathcal{G}(\beta, T) [\tilde{\mathcal{A}}(u) + \tilde{\mathcal{A}}(v)], \quad (49)$$

where $\mathcal{G}(\beta, T) = \sum_{n=1-2\beta}^{\infty} [(n+\beta)/(e^{(n+\beta)/T} - 1)]$ plays the role of a scale factor. Since $\mathcal{G}(1/2, T) > \mathcal{G}(0, T)$ (see Fig. 2), for $T > 0$ we have:

$$\mathcal{T}_T^{\text{mixed}} > \mathcal{T}_T^{\text{non-mixed}}, \quad (50)$$

where the superscript “mixed” means ND or DN cases, whereas “non-mixed” means NN or DD cases.

Note that if we set up two different cavities, one of them with non-mixed and the other one with mixed boundary conditions, both with a thermal field as the initial state with temperatures given, respectively, by T_1 and T_2 such that $\mathcal{G}(1/2, T_2) - \mathcal{G}(0, T_1) = -1/16$, we get for both cases the same initial configurations of energy densities in the static zone. Then for a same arbitrary law of motion for both cavities, they will exhibit the same time evolutions for the energy densities \mathcal{T} (and hence the same force acting on the boundaries). An interesting case occurs if we set up $T_1 \approx 0.34$ and $T_2 = 0$. In this case we

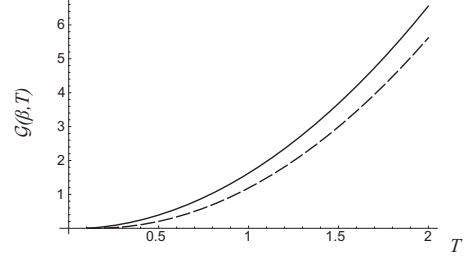


FIG. 2: Plot of $\mathcal{G}(\beta, T)$. The solid curve corresponds to $\mathcal{G}(1/2, T)$, whereas the dashed one to $\mathcal{G}(0, T)$.

obtain that the static and dynamical Casimir effect for the vacuum state in the mixed case (where the Casimir static force is repulsive) is mimicked by the non-mixed case with the mentioned specific temperature.

Let us now investigate a particular movement described by:

$$L(t) = L_0 \left[1 + \epsilon \sin \left(\frac{p\pi t}{L_0} \right) \right], \quad (51)$$

where $L_0 = 1$, $p = 2$ and ϵ is a dimensionless parameter. This oscillatory boundary motion is investigated in several papers (see, for instance, Refs. [9, 19]). In these papers the calculations were developed in the context of analytical approximate methods, considering small amplitudes of oscillation ($|\epsilon| \ll 1$). Here, we investigate the law of motion in Eq. (51) for $\epsilon = 0.01$ and $\epsilon = 0.1$. The former value is in better agreement with the condition $|\epsilon| \ll 1$ than the latter one. Our intention now is to verify, from an exact point of view, the similarity between the structures of \mathcal{T}_{vac} , $\mathcal{T}_{\text{non-vac}}$ and \mathcal{T} for both values of ϵ . In Fig. 3 we plot the ratio σ for the two values: $\epsilon = 0.01$ and $\epsilon = 0.1$. We observe that the ratio σ is more approximately constant for $\epsilon = 0.01$ (dashed line) than for $\epsilon = 0.1$ (solid line), so that, since $f^{(s)} + k_1 < 0$ and $g^{(s)} \leq 0$, we expect that \mathcal{T}_{vac} , $\mathcal{T}_{\text{non-vac}}$ and \mathcal{T} exhibit more similarity for the former than for the latter value of ϵ , for any diagonal initial state or any of the boundary conditions considered in the present paper. In fact, this is what can be visualized in Figs. 4 and 5, where we show \mathcal{T}_{vac} (dotted lines) and $\mathcal{T}_{\text{non-vac}}$ (solid lines) for the law of motion (51) and the thermal bath with temperature $T = 1$ as the diagonal initial field state. In Fig. 4, for the case $\epsilon = 0.01$, both dotted and solid lines have their minimum and maximum values at the same positions, as predicted by the approximate analytical approach considered by Andreata and Dodonov [19], which was based on $\epsilon \ll 1$. On the other hand, in Fig. 5 (case $\epsilon = 0.1$), if compared to the dotted line, the solid line has additional maximum points at $x = 0.28$ and $x = 0.72$, and also additional minimum points at $x = 0.42$ and $x = 0.58$, so that differences between the structures of \mathcal{T}_{vac} and $\mathcal{T}_{\text{non-vac}}$ start to become more evident.

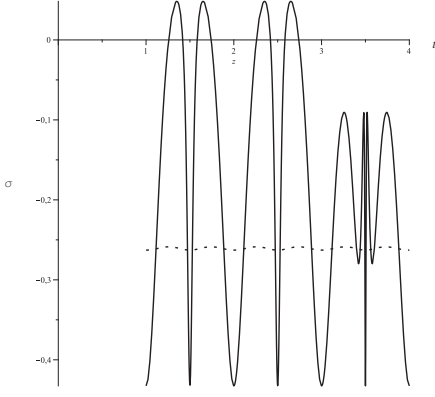


FIG. 3: The ratio $\sigma(z) = \tilde{\mathcal{B}}(z)/[\tilde{\mathcal{A}}(z)-1]$ for the law of motion given in Eq. (51). The dashed line corresponds to $\epsilon = 0.01$, whereas the solid line corresponds to $\epsilon = 0.1$.

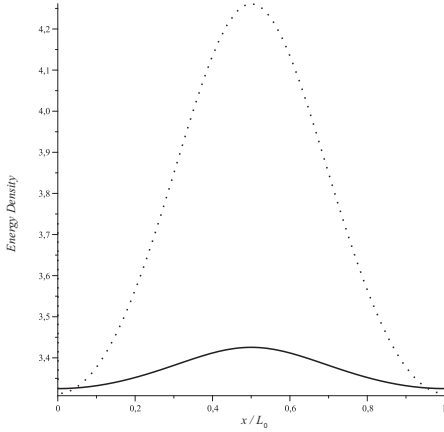


FIG. 4: The energy densities \mathcal{T}_{vac} (solid line) and $\mathcal{T}_{\text{non-vac}}$ (dotted line) at the instant $t = 1$, for both DD and NN cases. The law of motion considered is showed in Eq. (51), with the parameter $\epsilon = 0.01$. For the curve correspondent to $\mathcal{T}_{\text{non-vac}}$, it is being considered a the thermal bath with temperature $T = 1$ as the initial field state.

B. Non-diagonal states

When we consider non-diagonal states, the function $g^{(s)}$ in Eq. (40) can depend on $\tilde{z}(z)$ or, in other words, it can depend on the null line in the static zone where it is being calculated. Since Eq. (20) depends on γ^2 , now \mathcal{T} can be different for DD and NN cases (it is the same for DN and ND cases). We can see that \mathcal{T}_{vac} , $\mathcal{T}_{\text{non-vac}}$ and \mathcal{T} exhibit, in general, different structures.

The total energy stored in the cavity as a function of time can be written as:

$$\mathcal{E}(t) = -f^{(s)}\mathcal{F}_1(t) - \mathcal{F}_2(t) - \mathcal{F}_3(t), \quad (52)$$

where $\mathcal{F}_3(t) = \int_0^{L(t)} \left[g^{(s)}[\tilde{z}(u)]\tilde{\mathcal{A}}(u) + g^{(s)}[\tilde{z}(v)]\tilde{\mathcal{A}}(v) \right] dx$. The function \mathcal{F}_3 in Eq. (52) depends on the type of boundary condition, the initial field state and the law of motion considered. Next, we exemplify these situations

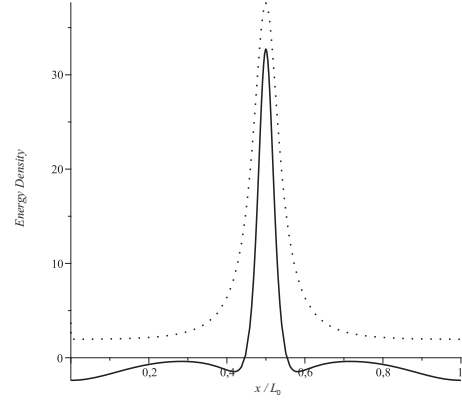


FIG. 5: The energy densities \mathcal{T}_{vac} (solid line) and $\mathcal{T}_{\text{non-vac}}$ (dotted line) at the instant $t = 1$, for both DD and NN cases. The law of motion considered is showed in Eq. (51), with the parameter $\epsilon = 0.1$. For the curve correspondent to $\mathcal{T}_{\text{non-vac}}$, it is being considered a the thermal bath with temperature $T = 1$ as the initial field state.

in the context of a coherent and a Schrödinger cat state, which are examples of non-diagonal initial field states.

The coherent state is defined as an eigenstate of the annihilation operator: $\hat{a}_n|\alpha\rangle = \alpha\delta_{nn_0}|\alpha\rangle$, where $\alpha = |\alpha|e^{i\theta}$ and n_0 is related to the frequency of the excited mode [27]. The Schrödinger cat state [28, 29] is defined as a superposition of coherent states $|\Psi\rangle = N(|\alpha\rangle + e^{i\phi}|- \alpha\rangle)$, where $|- \alpha\rangle$ has the same amplitude than $|\alpha\rangle$ but with a phase shift of π , and the normalization constant N is given by $N = (2 + 2e^{-2|\alpha|^2} \cos \phi)^{-1/2}$. We combine into a single formula the results for the energy density $\mathcal{T}_{\text{non-vac}}$ (now relabeled as \mathcal{T}_ζ) for both coherent and cat states, in the following manner:

$$\mathcal{T}_\zeta = \mathcal{T}_{(\hat{a}^\dagger \hat{a})_\zeta} + \mathcal{T}_{(\hat{a} \hat{a})_\zeta}, \quad (53)$$

where

$$\mathcal{T}_{(\hat{a}^\dagger \hat{a})_\zeta} = g_1^{(s)} \left[\tilde{\mathcal{A}}(u) + \tilde{\mathcal{A}}(v) \right], \quad (54)$$

$$\mathcal{T}_{(\hat{a} \hat{a})_\zeta} = g_2^{(s)} [\tilde{z}(u)]\tilde{\mathcal{A}}(u) + g_2^{(s)} [\tilde{z}(v)]\tilde{\mathcal{A}}(v), \quad (55)$$

with

$$g_1^{(s)} = \frac{|\gamma|^2 |\alpha|^2 \pi}{2L_0^2} (n_0 + \beta) \mathcal{C}(\phi, |\alpha|)^\eta, \quad (56)$$

and

$$g_2^{(s)}(z) = -\frac{|\alpha|^2 \gamma^2 \pi}{2L_0^2} (n_0 + \beta) \cos [2\pi(n_0 + \beta)z/L_0 - 2\theta], \quad (57)$$

where $\mathcal{C}(\phi, |\alpha|) = (1 - e^{-2|\alpha|^2} \cos \phi)/(1 + e^{-2|\alpha|^2} \cos \phi)$. For $\eta = 0$ we recover the coherent case, whereas for $\eta = 1$ the cat state energy density is obtained.

The two coherent states $|\alpha\rangle$ and $|- \alpha\rangle$ are not orthogonal to each other and their overlap $|\langle \alpha | - \alpha \rangle|^2 = e^{-4|\alpha|^2}$ decreases exponentially with $|\alpha|$. For $|\alpha|=2$ the overlap

is approximately zero [30], then we expect that the cat state behaves like two coherent states. From Eq. (53) for $|\alpha| \approx 2$ we have $\mathcal{C} \approx 1$ (see Fig. 6), hence the energy density for cat states and coherent states become the same. Both these energy densities are also the same, now for any value of $|\alpha|$, if we consider the Yurke-Stoler state (a cat state with phase $\phi = \frac{\pi}{2}$).

Observe that $g_1^{(s)}$ is constant in the static zone, whereas $g_2^{(s)}$ is spacetime dependent. From Eq. (41) and (54) we see that $\mathcal{T}_{\text{non-vac}}$ for diagonal states presents the same structure of $\mathcal{T}_{\langle \hat{a}^\dagger \hat{a} \rangle_\zeta}$. An interesting result occurs when one considers the cat state with $\phi = 0$ (odd cat state) and takes the limit $|\alpha| \rightarrow 0$ (leading to a single particle state) in Eq. (53). In this case we obtain $g^{(s)} = (\pi/2L_0^2)(n_0 + \beta)$ and $\mathcal{T}_{\langle \hat{a} \hat{a} \rangle_\zeta} = 0$. This means that the odd cat state in the limit $|\alpha| \rightarrow 0$ presents its energy density with the same structure of a single particle diagonal state as expected.

From Eq. (57) we see that θ contributes to the shape of the energy density in the static zone. The Fig. 7 shows the normalized coherent energy density $\mathcal{T}_\zeta^{(s)}|_{\eta=0}/|\alpha|^2$ as function of the position in a static situation, for fixed θ and several values of time and vice versa. Despite the fact that the energy density $\mathcal{T}_{\langle \hat{a} \hat{a} \rangle_\zeta}$ depends on θ , the total static energy $\mathcal{E}^{(s)}$ does not depend on this variable, as shown in Eq. (23). On the other hand, since $\mathcal{F}_3(t)$ can depend on θ , from Eq. (52) we see that the dynamical total energy \mathcal{E} can depend on this variable.

For different boundary conditions, the same configuration of energy density can be obtained by choosing different values of θ in each case. Note that under a phase shift $\theta \rightarrow \theta + \frac{\pi}{2}$ we have the following symmetries:

$$g_2^{(s) \text{ DD}}|_\theta = g_2^{(s) \text{ NN}}|_{\theta+\frac{\pi}{2}}, \quad g_2^{(s) \text{ DN}}|_\theta = g_2^{(s) \text{ ND}}|_{\theta+\frac{\pi}{2}}, \quad (58)$$

thus we get:

$$\mathcal{T}_\zeta^{(s) \text{ DD}}|_\theta = \mathcal{T}_\zeta^{(s) \text{ NN}}|_{\theta+\frac{\pi}{2}}, \quad \mathcal{T}_\zeta^{(s) \text{ DN}}(z)|_\theta = \mathcal{T}_\zeta^{(s) \text{ ND}}|_{\theta+\frac{\pi}{2}} \quad (59)$$

or, in other words, under the mentioned phase shift the energy densities for DN and ND (and also for DD and NN as shown in Fig. 7) in the static zone are the same. Then, from Eqs. (14) and (34) we obtain that the energy density at any instant t satisfies:

$$\mathcal{T}_\zeta^{\text{DD}}|_\theta = \mathcal{T}_\zeta^{\text{NN}}|_{\theta+\frac{\pi}{2}}, \quad \mathcal{T}_\zeta^{\text{DN}}|_\theta = \mathcal{T}_\zeta^{\text{ND}}|_{\theta+\frac{\pi}{2}}. \quad (60)$$

In Fig. 8 we show the normalized coherent energy density $\mathcal{T}_\zeta/|\alpha|^2$ for DD, NN, ND and DN cases, obtained via numerical calculations based on the exact Eqs. (54) and (55). In this figure, in which $\epsilon = 0.01$, the result for the DD case is in good agreement with the approximate analytical one obtained by Andreata and Dodonov for this boundary condition [19].

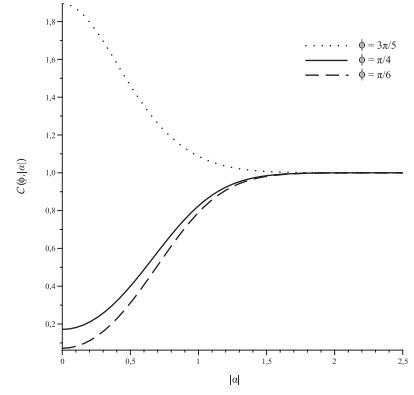


FIG. 6: $\mathcal{C}(\phi, |\alpha|)$ as function of $|\alpha|$, for different values of ϕ .

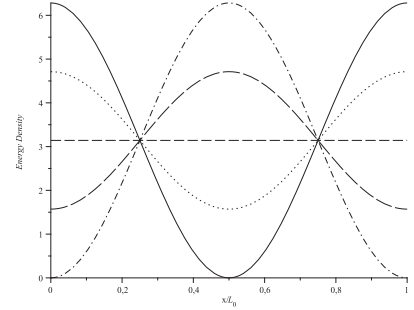


FIG. 7: The normalized energy density $\mathcal{T}_\zeta^{(s)}|_{\eta=0}/|\alpha|^2$ (coherent state) with $n_0 = 1$, as function of x/L_0 . The solid, dotted, dashed, dashed-dotted and long-dashed lines can have different meanings, for instance: (a) In the DD case for $t = 0$, they represent, respectively, the energy density for $\theta = 0, \pi/6, \pi/4, \pi/2$ and $\pi/3$; (b) In the DD case for $\theta = 0$, they represent the time evolution of the energy density in the static situation ($t < 0$), corresponding, respectively, to $t = 0, -1/6, -1/4, -1/2$ and $-1/3$. Also the correspondent NN situations can be obtained via Eq. (59).

V. FINAL COMMENTS

In summary, we extended to the case of an arbitrary initial field state the exact method proposed by Cole and Schieve [23], and also took into account the Neumann boundary condition. The formulas obtained in the present paper enable us to get exact numerical results for the energy density in a non-static cavity for an arbitrary law of motion. Moreover, the structure of the formulas are meaningful. For instance, they exhibit that, apart from the law of motion, the evolution of the energy density is completely determined by the total energy density in the static zone. In this sense we can set up two cavities with different boundary conditions and different initial field states, but in such manner that both have the same energy density in the static zone. In this situation we have (given the same law of motion for both cavities) the same time evolution for the energy density. We pointed that an interesting particular case occurs if we set up a non-mixed cavity with temperature 0.34 and a mixed

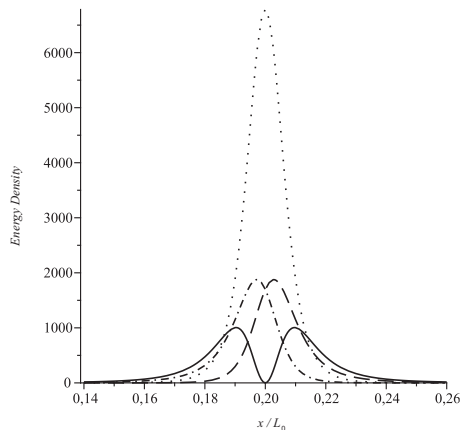


FIG. 8: The normalized coherent energy density $\mathcal{T}_\zeta/|\alpha|^2$ for $n_0 = 1$, $t = 50.3$ and the law of motion given in Eq. (51) with $\epsilon = 0.01$, as function of x/L_0 . The solid line corresponds to both: \mathcal{T}_ζ^{DD} with $\theta = 0$ and \mathcal{T}_ζ^{NN} with $\theta = \pi/2$. The dotted one corresponds to \mathcal{T}_ζ^{NN} with $\theta = 0$ and also \mathcal{T}_ζ^{DD} with $\theta = \pi/2$. The dashed-dotted line corresponds to \mathcal{T}_ζ^{DN} with $\theta = 0$ and also \mathcal{T}_ζ^{ND} with $\theta = \pi/2$. The dashed line shows \mathcal{T}_ζ^{ND} with $\theta = 0$ and also \mathcal{T}_ζ^{DN} with $\theta = \pi/2$.

cavity in the vacuum state. In this case we obtained that the static and dynamical Casimir effect for the vacuum state in a situation where the Casimir force is repulsive (the mixed case) are mimicked by another one for which the Casimir force is attractive (the non-mixed case) but with a certain non vanishing temperature.

We obtained that the energy densities \mathcal{T}_{vac} , $\mathcal{T}_{\text{non-vac}}$ and \mathcal{T} exhibit, in general, different structures. Wondering about the conditions under which they would present the same behavior, we found that this particular situation can occur for diagonal states and laws of motion for which the approximation given in Eq. (45) is valid. We showed that this condition is satisfied by the laws of motion investigated in Ref. [19], where the same structure for these energy densities was obtained. On the other hand we showed that if the condition (45) is not satisfied, these structures become different.

Focusing on the influence of the boundary conditions, we obtained that, for a same diagonal initial state and

(arbitrary) law of motion, the energy density for the DD case is identical to that obtained for the NN case, and the same occurs between ND and DN cases. This conclusion generalizes to any diagonal state the correspondent one obtained only for the vacuum case in Ref. [22], and also extends to the cavity problem the conclusion that the force acting on the moving boundary is invariant under the change $N \rightleftharpoons D$ when the initial field state is invariant under time translations (in Refs. [15, 17] this conclusion was obtained for the one-boundary case). Investigating the thermal field, we obtained specific formulas and concluded that, for a same temperature of the field state in the static zone, the energy density for the mixed case is bigger than the one obtained for the non-mixed case. For non-diagonal states, which have the energy density in the static zone non-invariant under time translation, we obtained that the energy density for the DD case in the dynamical situation is in general different from the one obtained for the NN case, and the same is valid between DN and ND boundary conditions. However, we showed that in special situations the equalities can be recovered. Investigating the coherent and Schrödinger cat states, we obtained specific formulas for these cases and verified that the difference between DD and NN cases (and also between ND and DN cases) is stored in $\mathcal{T}_{\langle \hat{a}\hat{a} \rangle_\zeta}$ part, whereas the $\mathcal{T}_{\langle \hat{a}^\dagger \hat{a} \rangle_\zeta}$ part behaves like the energy density of a diagonal state (all these conclusions are valid for any law of motion). We applied our method of calculation to the specific situation of a coherent state in a DD cavity with an oscillatory motion with small amplitude, verifying that the approximate result found in the literature for the energy density [19] is in good agreement with the one obtained via the exact method presented here. In addition, we also showed the correspondent results for NN, ND and DN cases.

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